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A study of a time varying paramter low pass filter circuit

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A STUDY OF A TIME-VARYING PARAMETER LOW PASS FILTER CIRCUIT

BY
H. A. Seymour and
N. Ederer

Thesis
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University of Pennsylvania

A STUDY OF A TIME VARYING PARAMETER LOW PASS FILTER CIRCUIT

By

H. A. Seymour

and

M. Ederer

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Submitted to the faculty of the
Moore School of the University of Pennsylvania in partial
fulfillment of the requirements for the degree of Master
of Science in Electrical Engineering

Philadelphia, May 1949

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Introduction

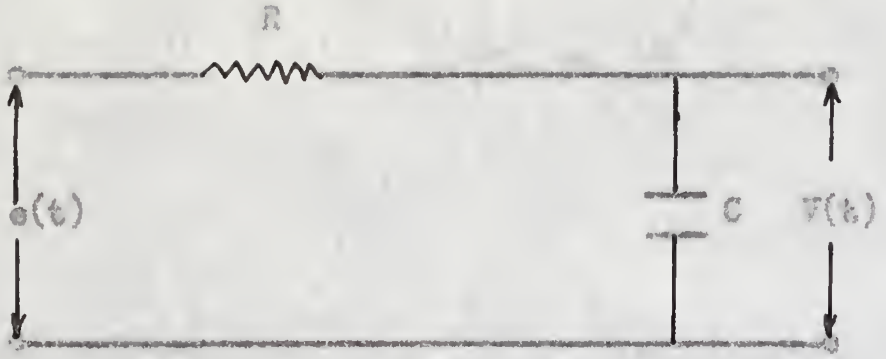

Fig. 1

The circuit shown in Fig. 1 above, is a low pass filter suitable for suppressing random noise appearing in the input voltage $e(t)$. Consider the input voltage $e(t)$ as composed of two parts:

1--A desirable signal input voltage of constant level (D.C.)*

2--Undesired noise, which varies rather rapidly with time.

By a suitable choice of resistance and capacitance it is possible to reduce the noise voltage appearing in the output $v(t)$ to a negligible level, while the desired signal is transmitted intact.

However, time varying signals present in the input signal in addition to the constant level will experience filtering to a smaller or greater extent alongside with the noise voltages, depending upon the rapidity of time variation.

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*It may perhaps seem rather unusual to regard a D.C. voltage or voltages varying linearly with time as signal voltages; one is accustomed to sinusoidal "low frequency signal" and "high frequency noise". However, constant, linearly varying or parabolically varying voltages are quite important in many servomechanism applications, principally in automatic fire control circuits.

In order to obtain an optimum compromise between the two conflicting requirements of maximum accuracy in signal reproduction and minimum noise output, it is sometimes necessary to vary the circuit parameters. The analysis of the circuit, after the variable parameter has assumed its final value, can easily be accomplished by conventional circuit theory; however, few investigations have been made in the past concerning the behavior of output voltage during transition time. In fact, past analyses have often borrowed the solutions valid for constant parameters, replacing the constant parameter by its time-varying equivalent. This approximate method of solution very often yields surprisingly accurate results; however, it breaks down in other cases.

It is the aim of this paper to investigate the exact performance of the resistance-capacitance series circuit with time varying resistance, and to check the range of error involved in the approximate method of evaluating the circuit behavior. The resistance variation to be considered primarily is a linear variation; higher order terms are considered in the appendix.

General Remarks

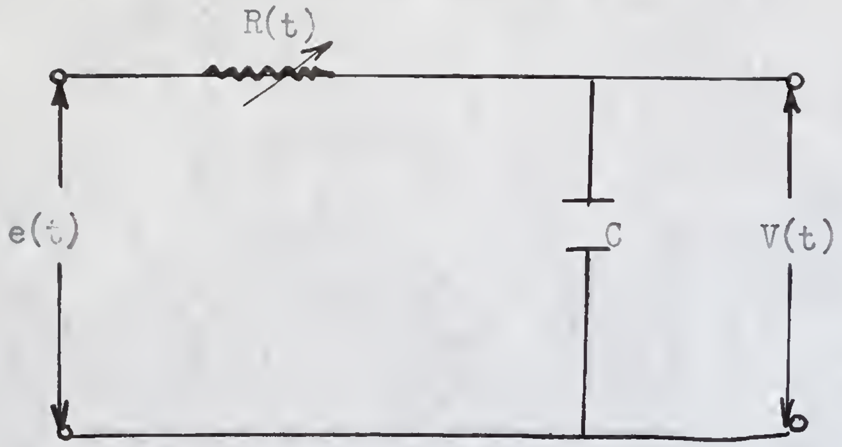

Fig. 2

The differential equation governing the above circuit is

(1) $CR(t)\frac{dv(t)}{dt} + v(t) = e(t)$

A linear variation of resistance is assumed; i.e.

(2) $R(t) = R_0 + R_1 t$

For convenience define

(3) $CR(t) = CR_0 + CR_1 t \equiv \tau_0 + \tau_1 t$

Two types of input voltages e(t) are to be studied, namely power series containing finite numbers of terms, and sinusoids. In many applications it is sufficiently accurate to regard the power series as merely a straight line or parabolic variation; emphasis will be placed on these two special cases. However, extension of the solution to any desired finite number of terms is given in the appendix.

Since the resistance variation discussed in this paper is a time function only, equation (1) is a linear equation; consequently the superposition principle may be applied. Thus, one may consider each term of e(t) by itself separately, and solve for the terms of v(t) which are due to this individual term of e(t) alone. As a final step the solutions obtained in this manner may be added, resulting in the solution due to all of the terms of e(t) combined.

Linearity and superposition are very convenient in this study, since they permit separate investigation of power series and sinusoidal signals. Furthermore, one is assured that neglection of higher order terms in e(t) will not affect the solutions of v(t) corresponding to the lower order terms of e(t).

Part I

Linear and Parabolic Signal Voltages

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The pertinent form of equation (1) is

$$(4)\quad (\tau_0 + \tau_1 t)\frac{dv(t)}{dt} + v(t) = \begin{cases} e_0 \text{ for constant signal voltage} \\ e_0 + e_1 t \text{ for linear signal voltage} \\ e_0 + e_1 t + e_2 t^2 \text{ for parabolic signal voltage} \end{cases}$$

Solutions to these differential equations are presented in Table I, subject to the boundary condition that at $t \leq 0$ the condenser is charged to the constant input voltage e_0; at $t = 0$ simultaneous variations in resistance and input signal begin. This boundary condition approximates the conditions of the practical applications of this problem.

The solutions for $\tau_1 = 0$ are also presented, since the previous approximate solution, presented in Table II, has merely replaced τ_0 by $\tau_0 + \tau_1 t$ wherever τ_0 appears in Table I, Col. 2.

The signal error $\varepsilon(t)$, defined as $v(t)-e(t)$ is tabulated in Table III as obtained by both the exact and approximate methods.

$\varepsilon(t)$ corresponding to $e(t) = e_0 + e_1 t$ is of immediate interest in design problems; inspection of Table III reveals that the signal error is independent of e_0; furthermore for the case of $e(t) = e_0 + e_1 t$ the signal error is directly proportional to e_1.

For plotting purposes it will be most convenient to define a new variable $\Phi \equiv -\frac{\varepsilon(t)}{e_1}$. Φ may perhaps be termed "per-unit signal error". Since the dimensions of $\varepsilon(t)$ and e_1 are volts and volts per second respectively, the proper dimensions of Φ are seconds.

Φ as calculated by both the exact and approximate methods is plotted against time with τ_0 and τ_1 as parameters. The following ranges of variables and parameters are presented as most nearly representative of automatic fire control applications:

1--Time: 0-20 seconds

2—τ_0: 1-5 seconds

3—τ_1: -.3 to .3

The fully drawn curves with index "Φex" represent solutions obtained by the exact method, while broken line curves with index "Φapp" represent solutions obtained by the approximate method. A single curve with index "Φex and Φapp" indicates that the differences between the two corresponding ordinates as calculated by the two methods are less than .1 unit of Φ(.1 second); this figure becomes comparable to the finite curve thickness; consequently a plot of the approximate curve was not warranted. However, should it become necessary to obtain more exact figures, especially in the neighborhood of t = 5 seconds, where differences between Φapp and Φex become inherently very small, one may resort to the curve entitled "Φapp - Φex vs. τ_1 at t = 5 seconds". Differences there are magnified in the ratio of 10:1.

The percentage error in output voltage $\eta \equiv \frac{\varepsilon(t)}{e(t)}$ is tabulated in Tables IV, V for $e(t) = e_0 + e_1 t$ at t = 10 seconds and 20 seconds respectively.

Since $\eta \equiv \frac{\varepsilon(t)}{e_0+e_1 t} = \frac{\frac{\varepsilon(t)}{e_1}}{\frac{e_0}{e_1} + t} = \frac{-\Phi}{\frac{e_0}{e_1}+t}$ it is possible to simultaneously tabulate $|\eta|$ for one positive and one negative ratio of $\frac{e_0}{e_1}$ at constant t. The upper (-) sign in the tabulations refers to the positive $\frac{e_0}{e_1}$ entered at the top of the column, and vice versa.

Table I- Exact Solutions

$e(t)$	$\vartheta(t)$	
	$CR(t) = T_0$	$CR(t) = T_0 + T_1 t$
e_0	e_0	e_0
$e_0 + e_1 t$	$[e_0 - T_0 e_1] + e_1 t + T_0 e_1 \varepsilon^{-\frac{t}{T_0}}$	$\left[e_0 - \frac{T_0 e_1}{1+T_1}\right] + \frac{e_1 t}{1+T_1}$ $+ \frac{T_0 e_1}{1+T_1}\left[\frac{T_0}{T_0+T_1 t}\right]^{\frac{1}{T_1}}$
$e_0 + e_1 t + e_2 t^2$	$[e_0 - T_0 e_1 + 2T_0^2 e_2]$ $+ [e_1 - 2T_0 e_2] t + e_2 t^2$ $+ [T_0 e_1 - 2T_0^2 e_2] \varepsilon^{-\frac{t}{T_0}}$	$\left[e_0 - \frac{T_0 e_1}{1+T_1} + \frac{2T_0^2 e_2}{(1+T_1)(1+2T_1)}\right]$ $+ \left[\frac{e_1}{1+T_1} - \frac{2T_0 e_2}{(1+T_1)(1+2T_1)}\right] t + \frac{e_2 t^2}{1+2T_1}$ $+ \left[\frac{T_0 e_1}{1+T_1} - \frac{2T_0^2 e_2}{(1+T_1)(1+2T_1)}\right]\left[\frac{T_0}{T_0+T_1 t}\right]^{\frac{1}{T_1}}$

Table II - Approximate Solutions

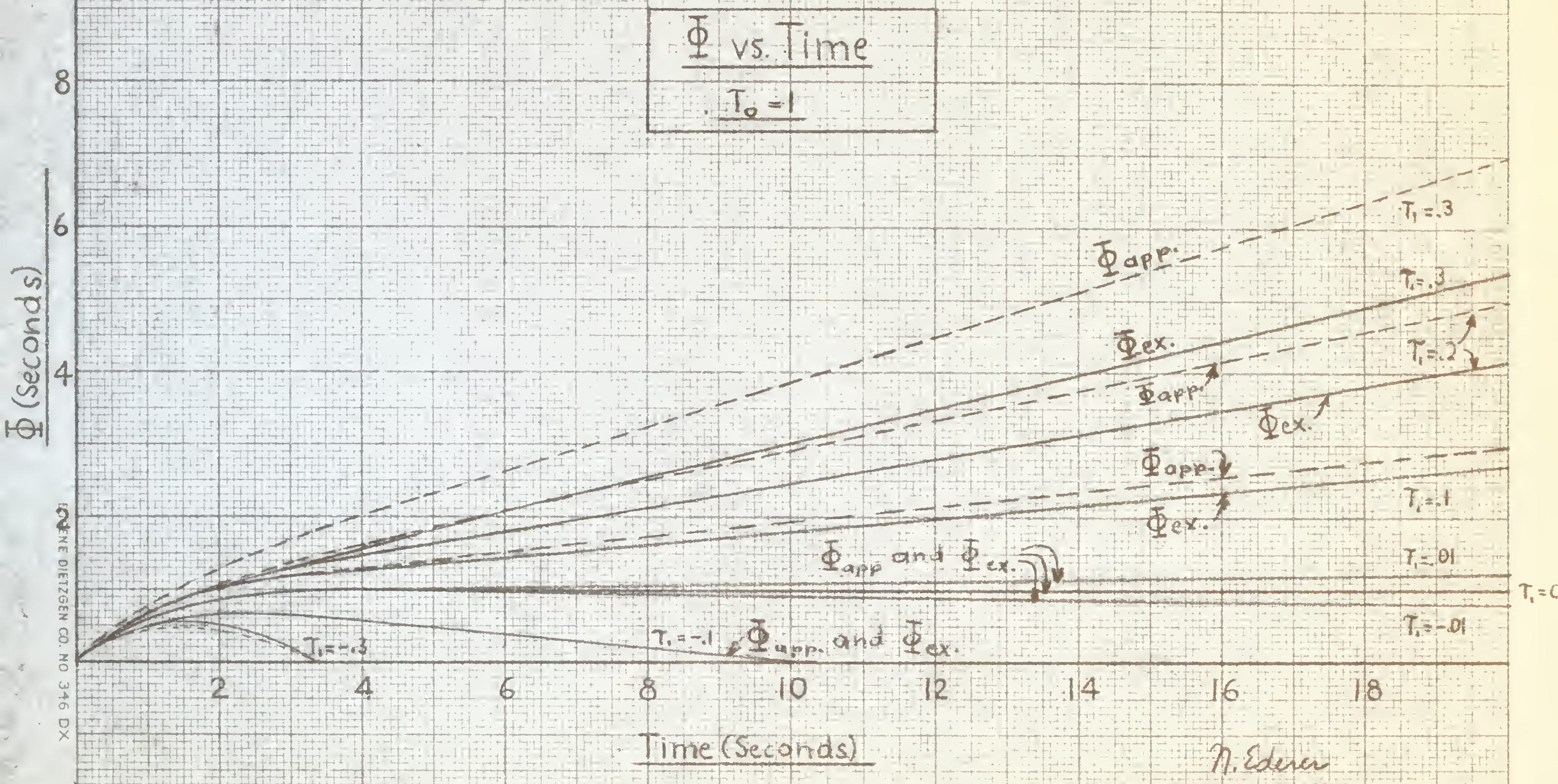
$CR(t) = T_0 + T_1 t$

$e(t)$	$v(t)$
e_0	e_0
$e_0 + e_1 t$	$[e_0 - T_0 e_1] + e_1 t [1 - T_1] + T_0 e_1 \varepsilon^{-\frac{t}{T_0 + T_1 t}}$
$e_0 + e_1 t + e_2 t^2$	$[e_0 - T_0 e_1 + 2T_0^2 e_2]$ $+ \left\{ e_1 [1 - T_1] + 2T_0 e_2 [2T_1 - 1] \right\} t$ $+ e_2 t^2 [1 - 2T_1 + 2T_1^2] + [T_0 e_1 - 2T_0^2 e_2] \varepsilon^{-\frac{t}{T_0 + T_1 t}}$

Table III - Signal Error $\varepsilon(t)$

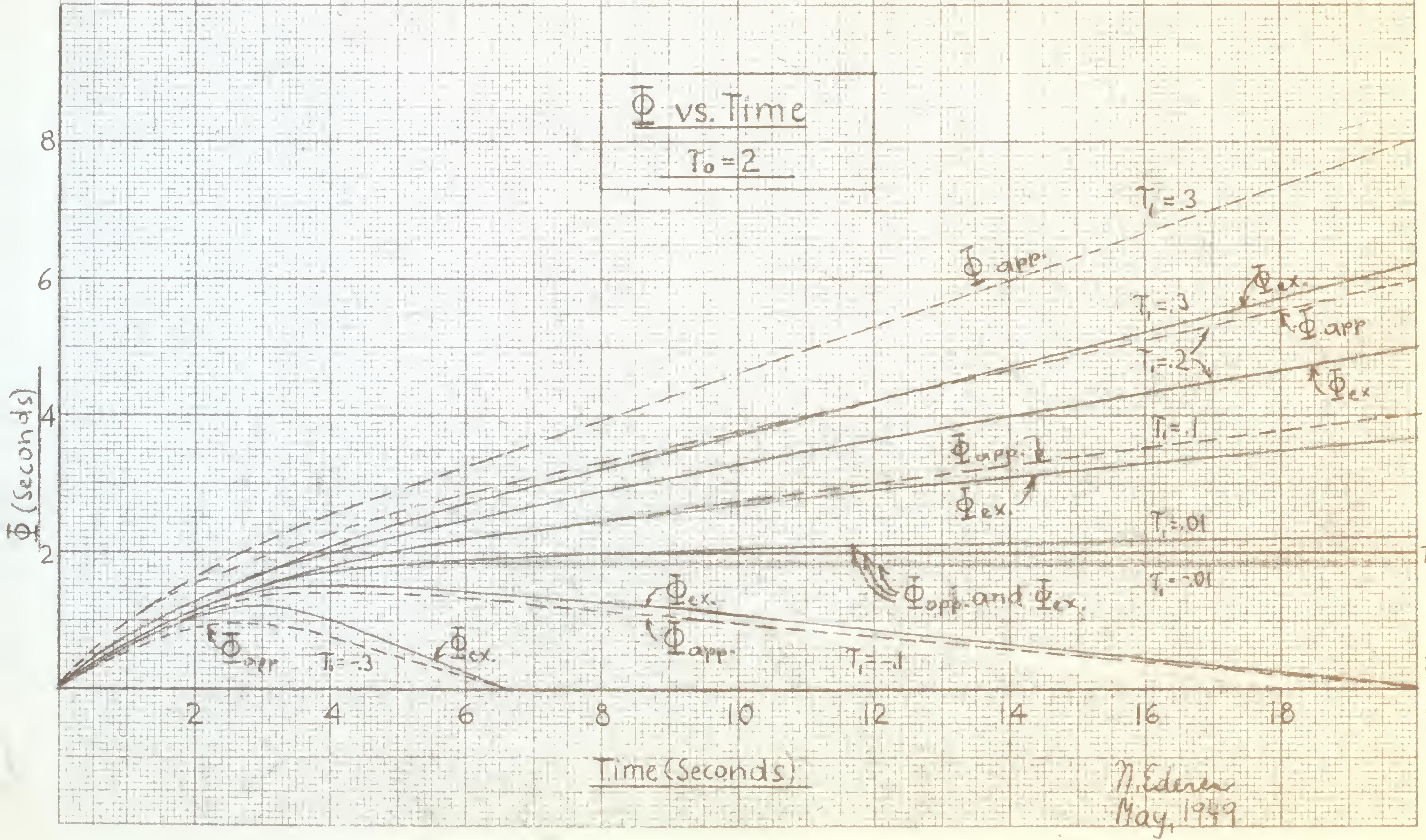
$(R(t) = T_0 + T_1 t)$

$e(t)$	$\varepsilon(t)$	
	Exact Solution $\varepsilon_{ex}(t)$	Approximate Solution $\varepsilon_{app}(t)$
e_0	None	None
$e_0 + e_1 t$	$-e_1\left[\frac{T_0}{1+T_1}\left\{1-\left(\frac{T_0}{T_0+T_1 t}\right)^{\frac{1}{T_1}}\right\}$ $+\frac{T_1 t}{1+T_1}\right]$	$-e_1\left[T_0\left\{1-\varepsilon^{-\frac{t}{T_0+T_1 t}}\right\}+T_1 t\right]$
$e_0 + e_1 t + e_2 t^2$	$\left[\frac{-T_0 e_1}{1+T_1}+\frac{2T_0^2 e_2}{(1+T_1)(1+2T_1)}\right]$ $-\left[\frac{T_1 e_1}{1+T_1}+\frac{2T_0 e_2}{(1+T_1)(1+2T_1)}\right]t$ $-\frac{2T_1 e_2 t^2}{1+2T_1}$ $+\left[\frac{T_0 e_1}{1+T_1}-\frac{2T_0 e_2}{(1+T_1)(1+2T_1)}\right]\left[\frac{T_0}{T_0+T_1 t}\right]^{\frac{1}{T_1}}$	$\left[-T_0 e_1 + 2T_0^2 e_2\right]$ $-\left[T_1 e_1 + 2T_0 e_2(1-2T_1)\right]t$ $+\left[T_1-1\right]2T_1 e_2 t^2$ $+\left[T_0 e_1 - 2T_0^2 e_2\right]\varepsilon^{\frac{t}{T_0+T_1 t}}$



EUGENE DIETZGEN CO. NO 346 DX

N. Ederer
May, 1949



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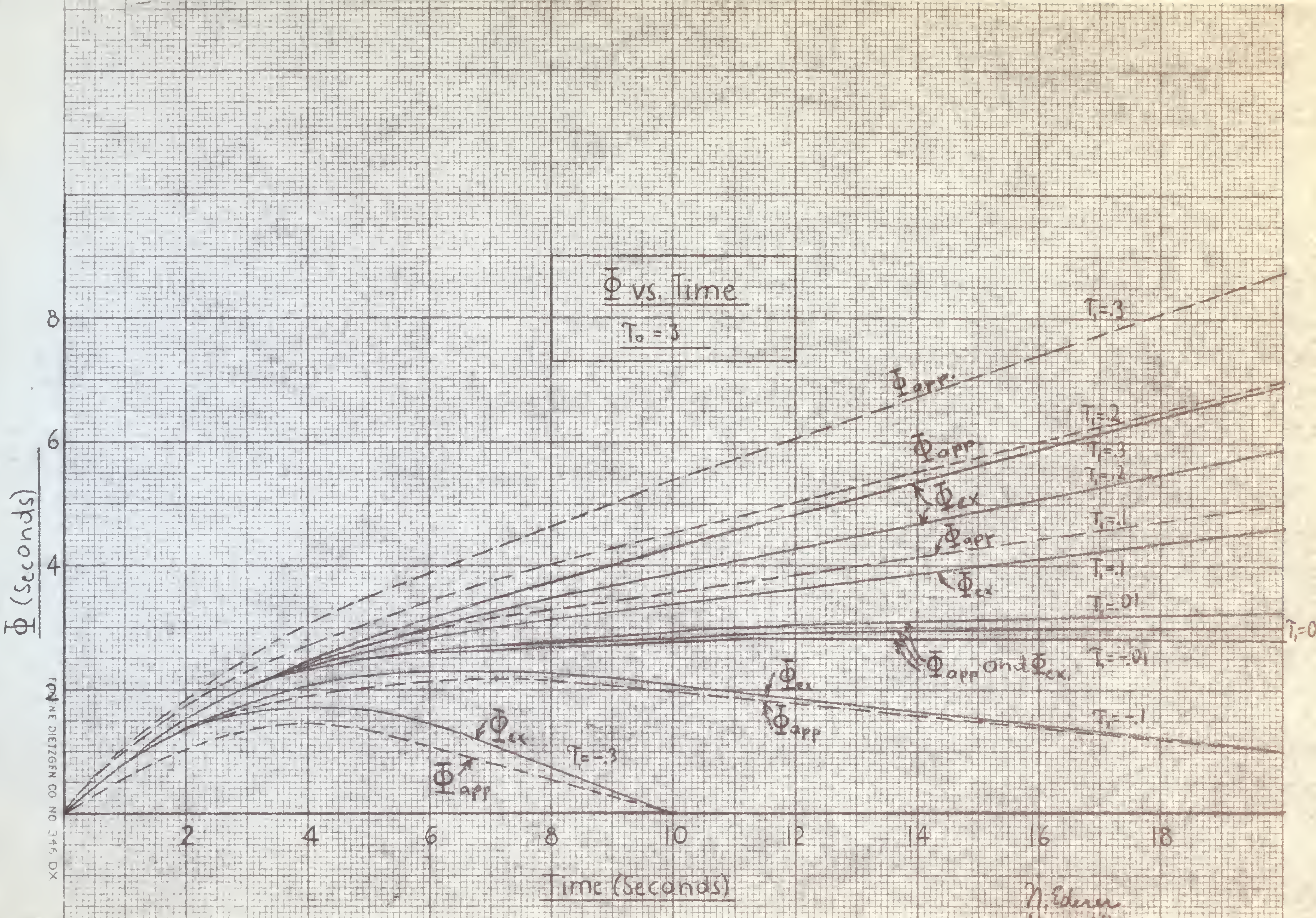

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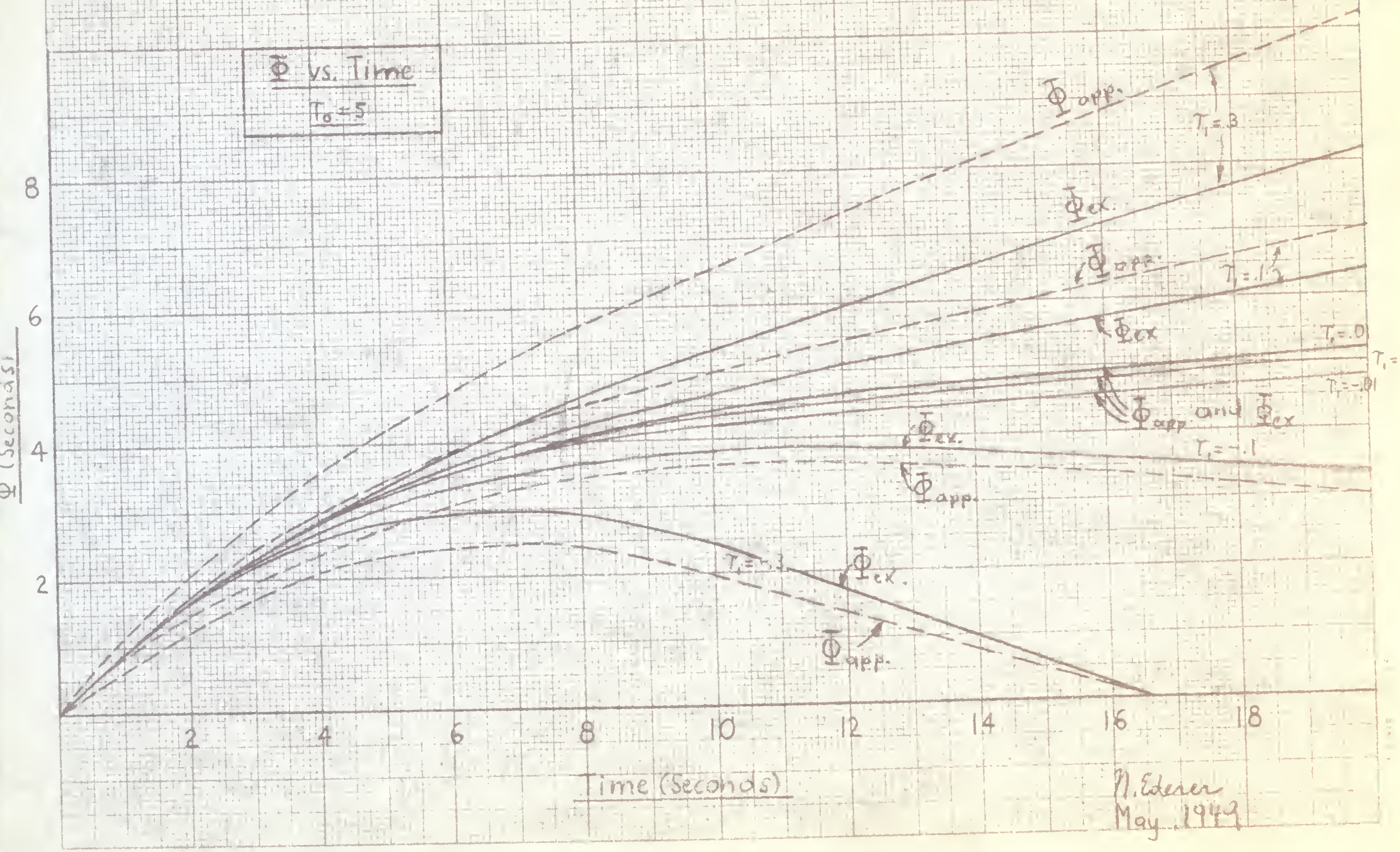
May 1949



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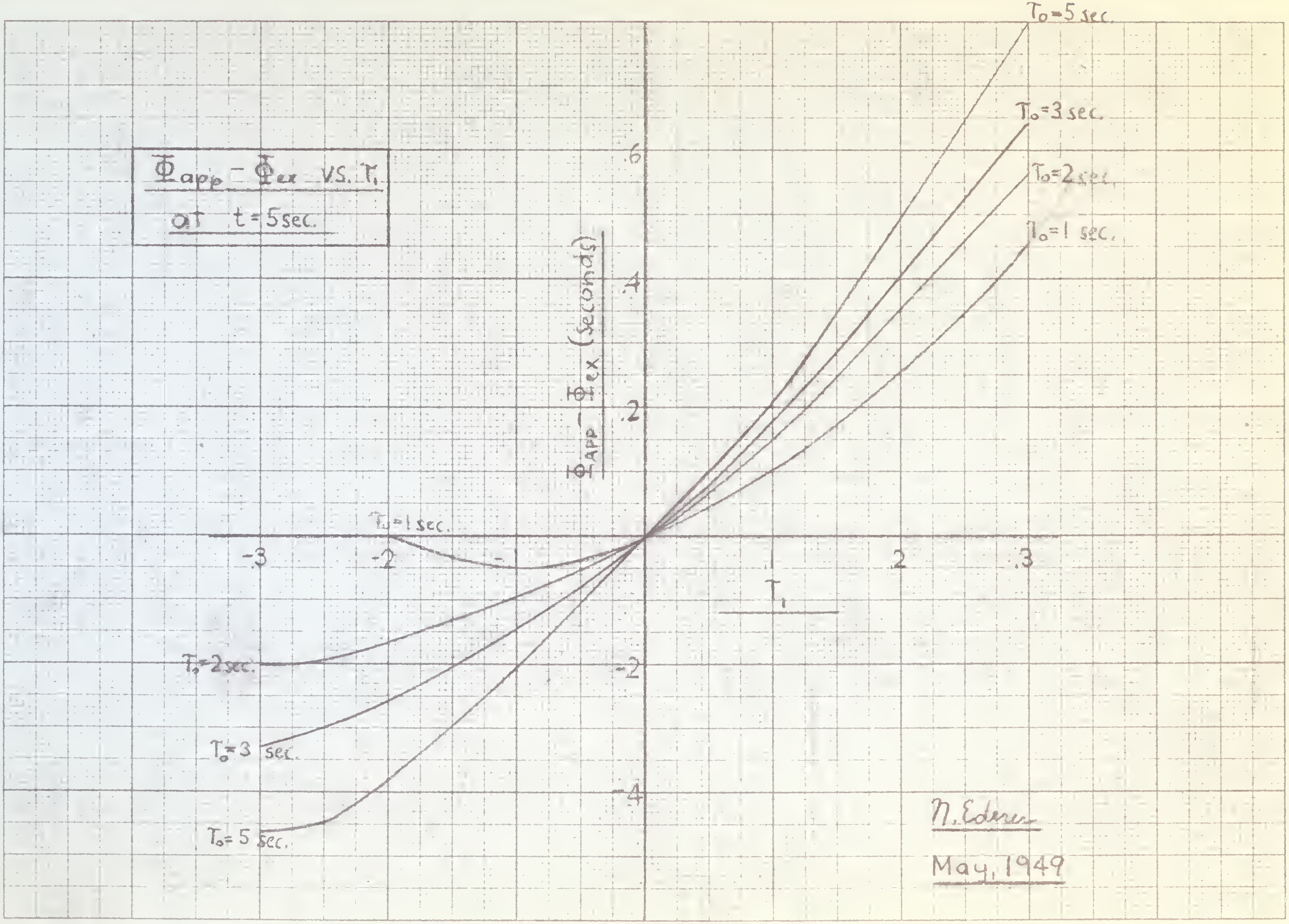
May 1949


$\Phi_{app} - \Phi_{ex}$ VS. T_1 at $t = 5$ sec.

N. Ederer
May, 1949

Table IV

Per cent Error in Output Voltage $\eta = \frac{\varepsilon(t)}{e(t)}$ at $t = 10$ sec.

$e(t) = e_0 + e_1 t$, $CR(t) = T_0 + T_1 t$

T_0 (sec)	T_1	$\frac{e_0}{e_1} = \begin{Bmatrix} +10 \\ -30 \end{Bmatrix}$ sec.			$\frac{e_0}{e_1} = \begin{Bmatrix} +40 \\ -60 \end{Bmatrix}$ sec.		
		η_{app} (%)	η_{ex} (%)	$\eta_{app} - \eta_{ex}$ (%)	η_{app} (%)	η_{ex} (%)	$\eta_{app} - \eta_{ex}$ (%)
1	.3	∓19.6	∓15.4	∓4.2	∓7.85	∓6.16	∓1.69
	.2	∓15.0	∓12.5	∓2.5	∓6.0	∓5.0	∓1.0
	.1	∓10.0	∓9.1	∓0.9	∓4.0	∓3.6	∓0.4
	-.1	∓0.0	∓0.0	∓0.0	∓0.0	∓0.0	∓0.0
2	.3	∓23.6	∓18.8	∓4.8	∓9.5	∓7.5	∓2.0
	.2	∓19.4	∓16.6	∓2.8	∓7.7	∓6.7	∓1.0
	.1	∓15.0	∓13.6	∓1.4	∓6.0	∓5.5	∓0.5
	-.1	∓5.0	∓5.6	±0.6	∓2.0	∓2.2	±0.2
3	.3	∓27.1	∓21.8	∓5.3	∓10.8	∓8.7	∓2.1
	.2	∓22.9	∓20.6	∓2.3	∓9.2	∓8.3	∓0.9
	.1	∓18.7	∓17.3	∓1.4	∓7.5	∓6.9	∓0.6
	-.1	∓10.0	∓10.5	±0.5	∓4.0	∓4.2	±0.2
	-.3	∓0.0	∓0.0	±0.0	∓0.0	∓0.0	±0.0
5	.3	∓32.9	∓26.8	∓6.1	∓13.1	∓10.7	∓2.4
	.1	∓25.2	∓23.6	∓1.6	∓10.1	∓9.4	∓0.7
	-.1	∓17.9	∓18.9	±1.0	∓7.2	∓7.6	±0.4
	-.3	∓10.0	∓12.0	±2.0	∓4.0	∓4.8	±0.8

Table IV - Continued

T_0 (sec)	T_1	$\frac{\ell_0}{\ell_1} = \begin{Bmatrix}+90\\-110\end{Bmatrix}$ sec.			$\frac{\ell_0}{\ell_1} = \begin{Bmatrix}+140\\-160\end{Bmatrix}$ sec.		
		η_{app} (%)	η_{ex} (%)	$\eta_{app}-\eta_{ex}$ (%)	η_{app} (%)	η_{ex} (%)	$\eta_{app}-\eta_{ex}$ (%)
1	.3	∓3.92	∓3.08	∓.84	∓2.61	∓2.05	∓.56
	.2	∓3.00	∓2.50	∓.50	∓2.00	∓1.67	∓.33
	.1	∓2.00	∓1.82	∓.18	∓1.33	∓1.21	∓.12
	-.1	∓0.00	∓0.00	∓0	∓0	∓0	0
2	.3	∓4.73	∓3.77	∓.96	∓3.15	∓2.51	∓.64
	.2	∓3.84	∓3.33	∓.51	∓2.56	∓2.22	∓.34
	.1	∓3.00	∓2.73	∓.27	∓2.00	∓1.82	∓.18
	-1	∓1.00	∓1.01	±.01	∓.67	∓.74	±.07
3	.3	∓5.43	∓4.37	∓1.06	∓3.62	∓2.41	∓.71
	.2	∓4.69	∓4.13	∓.56	∓3.06	∓2.75	∓.31
	.1	∓3.75	∓3.47	∓.28	∓2.50	∓2.31	∓.19
	-.1	∓2.00	∓2.09	±.09	∓1.33	∓1.39	±.06
	-.3	∓0.00	∓0.00	∓0	∓0.00	∓0.00	0
5	.3	∓6.57	∓5.35	∓1.22	∓4.38	∓3.57	∓.81
	.1	∓5.05	∓4.71	∓.34	∓3.36	∓3.14	∓.22
	-.1	∓3.58	∓3.79	±.21	∓2.38	∓2.52	±.14
	-.3	∓2.00	∓2.41	±.41	∓1.33	∓1.60	±.27

Table IV - Continued

T_0 (sec.)	T_1	$\frac{\ell_2}{\ell_1} = \begin{Bmatrix}+190\\-210\end{Bmatrix}$ sec.			$\frac{\ell_2}{\ell_1} = \begin{Bmatrix}+240\\-260\end{Bmatrix}$ sec.			$\frac{\ell_2}{\ell_1} = \begin{Bmatrix}+290\\-310\end{Bmatrix}$ sec.		
		η_{app} (%)	η_{ex} (%)	$\eta_{opp}-\eta_{ex}$ (%)	η_{opp} (%)	η_{ex} (%)	$\eta_{app}-\eta_{ex}$ (%)	η_{app} (%)	η_{ex} (%)	$\eta_{app}-\eta_{ex}$ (%)
1	.3	∓1.96	∓1.58	∓.41	∓157	∓123	∓34	∓1.30	∓1.03	∓.27
	.2	∓1 50	∓125	∓.25	∓120	∓1.00	∓20	∓100	∓.84	∓.16
	.1	∓1.00	∓.91	∓.09	∓.80	∓.73	∓.07	∓.67	∓61	∓.06
	-.1	∓0.00	∓000	0	∓0.00	∓000	0	∓0.00	∓0.0	0
2	.3	∓2.36	∓188	∓.78	∓1.89	∓1.50	∓.49	∓1.57	∓1.25	∓32
	.2	∓1.94	∓1.67	∓.27	∓1.54	∓1.33	∓.21	∓1.28	∓1.11	∓.17
	.1	∓150	∓1.36	∓.14	∓1.20	∓1.09	∓.11	∓1.00	∓.91	∓09
	-1	∓.50	∓.52	±.02	∓.40	∓.44	±.04	∓33	∓.37	±04
3	.3	∓271	∓2.18	∓53	∓2.17	∓1 75	∓.42	∓1.81	∓146	∓.35
	.2	∓2.29	∓2.07	∓22	∓1.83	∓1.65	∓.18	∓153	∓137	∓.16
	.1	∓187	∓1.73	∓14	∓1.50	∓1.38	∓.12	∓1.25	∓1.15	∓.10
	-1	∓100	∓104	±.04	∓.80	∓.84	±.04	∓.67	∓.70	±.03
	-3	∓0 00	∓0.00	0	∓0.00	∓0.00	∓0.0	∓0.00	∓0.00	0
5	3	∓3.29	∓2.68	∓.61	∓2 63	∓2.14	∓.49	∓219	∓1.78	∓.41
	.1	∓2 52	∓2.36	∓.16	∓2 02	∓1.88	∓.14	∓168	∓158	∓.10
	-1	∓1.79	∓1.94	±.15	∓1.43	∓1.51	±.08	∓119	∓1.26	±.07
	-3	∓100	∓1.20	±.20	∓.80	∓.96	±.16	∓67	∓80	±.13

Table V

Percent Error in Output Voltage $\eta = \frac{\varepsilon(t)}{e(t)}$ at $t = 20$ sec

$e(t) = e_0 + e_1 t$; $CR(t) = \tau_0 + \tau_1 t$

τ_0 (sec.)	τ_1	$\frac{e_0}{e_1} = \begin{Bmatrix} +30 \\ -70 \end{Bmatrix}$ sec.			$\frac{e_0}{e_1} = \begin{Bmatrix} +80 \\ -120 \end{Bmatrix}$ sec		
		η_{app} (%)	η_{ex} (%)	$\eta_{app} - \eta_{ex}$ (%)	η_{apr} (%)	η_{ex} (%)	$\eta_{app} - \eta_{ex}$ (%)
1	.3	∓140	∓10.78	∓3.22	∓700	∓5.39	∓.61
	.2	∓10.0	∓832	∓1.68	∓5.00	∓4.16	∓.84
	.1	∓6.0	∓5.46	∓.54	∓3.00	∓2.73	∓.27
2	.3	∓16.0	∓12.32	∓3.68	∓8.00	∓6.16	∓1.84
	.2	∓12.0	∓10.00	∓2.00	∓6.00	∓5.00	∓1.00
	.1	∓8.0	∓7.28	.72	∓4.00	∓3.64	∓.36
	-1	∓0.0	∓0.0	0	∓0.00	∓0.0	0
3	.3	∓17.36	∓1368	∓3.68	∓8.68	∓6.63	∓1.95
	.2	∓1400	∓11.66	∓234	∓7.00	∓5.42	∓1.58
	.1	∓10 00	∓9.08	∓.92	∓5.00	∓4.48	∓.52
	-.1	∓2.00	∓2.22	± .22	∓1.00	∓1.11	± .11
5	.3	∓20.40	∓16.36	∓4.04	∓10.20	∓8.18	∓2.02
	.1	∓14.00	∓12.72	∓1.28	∓7.00	∓6.36	∓.64
	-.1	∓6.00	∓6.66	± .66	∓3.00	∓3.33	± .33

Table V - Continued

T_0 (sec)	τ_1	$\frac{\ell_2}{\ell_1} = \begin{Bmatrix}+130\\-170\end{Bmatrix}$ sec.			$\frac{\ell_0}{\ell_1} = \begin{Bmatrix}+180\\-220\end{Bmatrix}$ sec		
		η_{app} (η_0)	η_{ex} (η_0)	$\eta_{app}-\eta_{ex}$ (η_0)	η_{app} (η_0)	η_{ex} (η_0)	$\eta_{app}-\eta_{ex}$ (η_0)
1	.3	∓466	∓360	∓106	∓350	∓269	∓81
	.2	∓3.33	∓2.78	∓55	∓250	∓208	∓42
	1	∓200	∓182	∓18	∓1.50	∓1.36	∓14
2	3	∓533	∓412	∓121	∓400	∓318	∓82
	2	∓400	∓333	∓67	∓300	∓250	∓50
	.1	∓2.66	∓2.43	∓.23	∓2.00	∓132	∓18
	-.1	∓000	∓000	0	∓0.00	∓000	0
3	.3	∓5.78	∓4.56	∓122	∓4.34	∓3.37	∓.97
	.2	∓466	∓389	∓.77	∓350	∓288	∓.62
	1	∓333	∓303	∓30	∓250	∓224	∓.26
	-1	∓.66	∓.74	±.12	∓50	∓.55	±05
5	3	∓680	∓545	∓135	∓510	∓404	∓101
	1	∓4.66	∓425	∓.41	∓3.50	∓313	∓.32
	-.1	∓200	∓222	±.22	∓1.50	∓167	±17

Table V - Continued

T_0 (sec)	T_1	$\frac{\ell_2}{\ell_1} = \left\{\begin{matrix}+230\\-270\end{matrix}\right\}$ sec			$\frac{\ell_2}{\ell_1} = \left\{\begin{matrix}+280\\-320\end{matrix}\right\}$ sec			$\frac{\ell_2}{\ell_1} = \left\{\begin{matrix}+330\\-370\end{matrix}\right\}$ sec		
		η_{app} (%)	η_{ex} (%)	$\eta_{app}-\eta_{ex}$ (%)	η_{app} (%)	η_{ex} (%)	$\eta_{app}-\eta_{ex}$ (%)	η_{app} (%)	η_{ex} (%)	$\eta_{app}-\eta_{ex}$ (%)
1	.3	∓2.80	∓2.15	∓.65	∓2.33	∓1.79	∓.54	∓2.00	∓1.54	∓.46
	.2	∓2.00	∓1.66	∓.34	∓1.66	∓1.39	∓.27	∓1.43	∓1.19	∓.24
	.1	∓1.20	∓1.09	∓.11	∓1.00	∓.91	∓.09	∓.86	∓.78	∓.08
2	.3	∓3.20	∓2.46	∓.74	∓2.67	∓2.05	∓.62	∓2.28	∓1.76	∓.52
	.2	∓2.40	∓2.00	∓.40	∓2.00	∓1.66	∓.34	∓1.71	∓1.43	∓.28
	.1	∓1.60	∓1.45	∓.15	∓1.33	∓1.21	∓.12	∓1.14	∓1.04	∓.10
	-.1	∓0.00	∓0.00	0	∓0.00	∓0.00	0	∓0.00	∓0.00	0
3	.3	∓3.49	∓2.73	∓.76	∓2.89	∓2.28	∓.61	∓2.48	∓1.95	∓.53
	.2	∓2.80	∓2.33	∓.47	∓2.33	∓1.94	∓.39	∓2.00	∓1.67	∓.33
	.1	∓2.00	∓1.78	∓.22	∓1.67	∓1.51	∓.16	∓1.43	∓1.30	∓.13
	-.1	∓.40	∓.44	±.04	∓.33	∓.37	±.04	∓.29	∓.32	±.03
5	.3	∓4.08	∓3.27	∓.81	∓3.40	∓2.72	∓.68	∓2.92	∓2.34	∓.58
	.1	∓2.80	∓2.54	∓.26	∓3.33	∓2.12	∓.21	∓2.00	∓1.82	∓.18
	-.1	∓1.20	∓1.33	±.11	∓1.00	∓1.11	±.11	∓.86	∓.95	±.09

Conclusions

In design work the following considerations may be important:

1--Absolute magnitude of error $\varepsilon(t) = v(t) - e(t)$

2--Percent magnitude of error $\eta = \frac{\varepsilon(t)}{e(t)}$

Two methods of calculating $\varepsilon(t)$ and η are available, namely the exact and approximate methods.

For linear signal voltage and linear resistance variation, the graphical and tabular results offer quick means to compute $\varepsilon(t)$ and η.

To find $\varepsilon(t)$ corresponding to given Γ_o, Γ_1 and time, it is merely necessary to multiply the appropriate ordinate by $-e_1$. This may be done graphically by adjusting the scale of ordinates to the required e_1, recalling that $\Phi = -\frac{\varepsilon(t)}{e_1}$.

The absolute magnitude of the error becomes greater with time, Γ_o, Γ_1 and e_1. For positive Γ_1 the approximate absolute error becomes progressively greater than the exact error, as each of the aforementioned parameters become larger; however, the opposite is true for negative Γ_1.

Thus the maximum error corresponds to $\Gamma_o = 5$ seconds, $\Gamma_1 = 3$, and $t = 20$ seconds for which $\varepsilon ex(20) = 8.2\ e_1$ volts, $\varepsilon app(20) = -10.3\ e_1$ volts, a difference of $-2.1\ e_1$ volts. On the other hand a more accurate plot of the difference at $t = 5$ sec. shows that $\varepsilon app(5) - \varepsilon ex(5)$ is $-.8\ e_1$ volts corresponding to $\Gamma_o = 5$ sec., $\Gamma_1 = .3$.

The percent error η depends on the additional parameter $\frac{e_o}{e_1}$. Inspection of Tables IV and V indicates that for a wide range of $\frac{e_o}{e_1}$ at t = 10 sec. and t = 20 sec. The difference between η_{app} and η_{ex} does not exceed 5%. To reduce η to a minimum $\frac{e_o}{e_1}$ should be high.

Ranges of linear signal voltages not plotted or tabulated as well as parabolic voltages may be investigated with the aid of the expressions of Tables I, II, III. It is hoped that sufficient preliminary investigation has been made concerning higher order voltage and resistance terms, as presented in the Appendix. The following limitations arise in the considerations of higher order terms:

1--The number of parameters increases to a point where general calculations become very cumbersome; for any given set of values, however, calculations are fairly simple.

2--Some of the solutions obtained there are of necessity presented in integral form; it would probably be necessary to perform these integrations on the differential analyzer.

Part II

Sinusoidal Signal Voltages

Solution of the Differential Equation

For CR(t) of the form $\tau_o + \tau_1 t$, and excluding the case for $\tau_1 = 0$ which has been solved elsewhere in this paper, the solution of the differential equation for sinusoidal signal voltages requires the solution of the following integral equation:

$$(5)\ V = \left[K_2 + \int (A \cos \omega t + B \sin \omega t)\ (\tau_o + \tau_1 t)^{\frac{1}{\tau_1} - 1}\ dt \right]\ (\tau_o + \tau_1 t)^{-\frac{1}{\tau_1}}$$

We can set the arbitrary constant K_2 equal to V(o) in which case we have our integral equation in the following form:

$$(6)\ V = \left[\ V(o) + \int_o^T \frac{A \cos\ \omega t + B \sin \omega t}{\tau_o + \tau_1\ t}\, e^{\int_o^T \frac{dt}{\tau_o + {}_1 t}}\ dt \right]\ e^{-\int_o^T \frac{dt}{\tau_o + \tau_1 t}}$$

When the exponent $\frac{1}{\tau_1} - 1$ is a small positive integral number, we are able to evaluate the integral directly by the process of expansion of the term $(\tau_o + \tau_1 t)^{\frac{1}{\tau_1} - 1}$. Solutions are tabulated for $\tau_1 = 1, 2, 3,$ and 4, in addition to the case in which $\tau_1 = 0$ which is the constant parameter case. The solutions are tabulated for the general case in which A and B have non-zero values; this is the case for any phase angle of the composite input wave. Solutions are also tabulated for the case in which A = 0, corresponding to a phase angle of $\frac{\pi}{2}$ and for the case in which B = 0, corresponding to a phase angle of zero.

For higher positive integer values of $\frac{1}{\tau_1}$, for negative integer values of $\frac{1}{\tau_1}$, and for all fractional values of $\frac{1}{\tau_1}$ recourse must be had to a differential analyzer, a Fourier coefficient analyzer, or numerical hand computation. In attacking the problem, however, we are particularly interested in the range $-.3 < \tau_1 < .3$ and it is to be noted that the tabulated solutions for $\tau_1 = 0$ (special case) and $\tau_1 = .25$ fall in this range. Further, the case wherein $\tau_1 = .33$ lies at the extreme positive end of the range.

Differential equation: $\frac{dV(t)}{dt} + \frac{V(t)}{CR(t)} = \frac{e(t)}{CR(t)}$

Signal voltage: $e(t) = A \cos \omega t + B \sin \omega t$

$$CR(t): \left\{ \begin{array}{ll} \text{Case I:} & CR(t) = \text{constant} = \tau_o \\ \text{Case II:} & CR(t) = \tau_o + \tau_1 t \end{array} \right.$$

$$\left.\begin{array}{l}\text{Steady state solution} \\ \text{for Case I:}\end{array}\right\} \text{(7)} \quad V_{\substack{\text{steady} \\ \text{State}}} = \frac{A - \omega\tau_o B}{\omega^2\tau_o^2 + 1} \cos\omega t + \frac{\omega\tau_o A + B}{\omega^2\tau_o^2 + 1} \sin\omega t$$

In the solutions which follow for Case II, the solutions are tabulated for the general case, for the case in which $A = 0$, and for the case in which $B = 0$.

Since $e(t) = A \cos \omega t + B \sin \omega t$:

$$e(t) = \sqrt{A^2 + B^2} \cos (\omega t - \theta)$$

where $\theta = \tan^{-1} \frac{B}{A}$

Hence, in the case where $A = 0$,

$$\theta = \frac{\pi}{2} \text{ and } e(t) = B \sin \omega t$$

In the case when $B = 0$,

$$\theta = 0 \text{ and } e(t) = A \cos \omega t$$

For 0 any value the general solution applies, in response to the signal voltage $e(t) = A \cos \omega t + B \sin \omega t$.

In order to indicate the variation of V as a function of τ_1, for small positive values of τ_1, the accompanying curves have been plotted under the following conditions:

$\gamma_1 = 0, 1/3, 1/2, 1$

$A = B = 1$

$\omega = 1$

$\gamma_0 = 1$

The solutions then become:

$$\gamma_1 = 0: \quad V = \sin t$$

$$\gamma_1 = 1/3: \; V = \frac{3}{(3+t)^3}\left[1 + (t^2+8t+13)\sin t - (t^2+4t+1)\cos t\right]$$

$$\gamma_1 = 1/2: \; V = \frac{2}{(2+t)^2}\left[1 + (3+2t)\sin t - (1+t)\cos t\right]$$

$$\gamma_1 = 1: \quad V = \frac{1}{1+t}\left[1 + \sin t - \cos t\right]$$

For γ_1 equal to zero, which is the case when $Gk(t)$ = constant = γ_0, we have a constant coefficient differential equation to solve, the signal voltage being a sum of sinusoids which may resolved into a single sinusoid with an initial phase angle. As is to be expected the solution for the response is a constant amplitude sinusoid. As γ_1 takes on small positive integer values the solutions become more complicated but, as can be seen from the equations and the curves, the response again involves a sinusoidal form, though now we have a vertical displacement of the sinusoid due to the appearance of the transient term which was neglected in the case where $\gamma_1 = 0$. In this latter case it was considered that at the starting of problem time the transient had already died out and steady state conditions existed. For small positive integer values of γ_1 the sinusoidal terms are modified by amplitudes that are not constant but that are functions of time. The net result is that the responses to sinusoidal inputs, in the case of CR(t) variable, are of a damped sinusoidal form with the damping factor being variable and not following a definite pattern. In general, increasing τ_1 decreases the amplitude of the response (an exception is seen in the case of $\tau_1 = \frac{1}{2}$ where the initial swing of the response is to a value greater than unity) though to make this comparison it is necessary to take into account the phase shift or expansion of the resultant cycle introduced by increasing τ_1 from its zero value. Decreasing τ_1 towards zero has the effect of changing the solution towards the constant amplitude sinusoidal form.

To give an indication of the effect of changing frequency the solutions have been plotted when only one input sinusoid is present for two values of angular frequency, $\omega = 1$ and $\omega = 2$. All other parameters have the same values for the two solutions. The curves show that as the frequency is increased the irregularly damped sinusoidal form responses are decreased in amplitude and die out sooner.

Again confining our attention to sinusoidal signal voltages the question comes up as to whether or not we are justified in substituting the actual variation of CR(t) into the solution for CR(t) = constant, and to what extent error is introduced by this procedure.

Our input signal voltage is $e(t) = A \cos \omega t + B \sin \omega t$. To obtain an average indication, consider a given solution in two ways; first, let $A = 0$, $B = 1$ which corresponds to $\theta = \frac{\pi}{2}$ and second, let $A = 1$, $B = 0$ which corresponds to $\theta = 0$. Denoting the first solution by V_B and the second by V_A, we will plot $V_A^2 + V_B^2$ for both the exact method of solution and for the approximate method wherein we substitute the actual variation of CR(t) in place of CR(t) = constant. Considering all other parameters fixed and the same for all solutions, we have:

For $\tau_1 = 1$:

Approximate: $V_A^2 + V_B^2 = \dfrac{1}{(1 + t)^2 + 1}$

Exact: $V_A^2 + V_B^2 = \dfrac{1.5 + \sin t - \cos t}{(1 + t)^2}$

For $\tau_1 = \frac{1}{2}$:

Approximate: $V_A^2 + V_B^2 = \dfrac{1}{(1 + .5\ t)^2 + 1}$

Exact: $V_A^2 + V_B^2 = \dfrac{(1 + .5t)^2 + .5 + .5 \sin t - (1 + .5t) \cos t}{(1 + .5t)^4}$

The curves show that the responses die out more slowly as τ_1 is decreased toward zero, and that agreement between the exact and approximate solutions is better for smaller τ_1. That the response should die out more slowly as τ_1 is reduced is to be expected since when $\tau_1 = 0$, the sum of the squares of V_A and V_B will be equal to a constant. The disparity between the approximate and the exact solutions as a function of τ_1 is another question. Obviously, at some value of τ_1 and for lower values, we will be within the limits of error specified by a given design problem if we use the approximate solution. This point must be determined by the situation to be met. Further, the justification of the use of the approximate solution cannot be found entirely in the behavior of τ_1; it is also a function of the values assumed by τ_0 and ω. Another point to be understood is that although plots of the values of V_A^2 plus V_B^2 are shown for values of time between zero and five seconds in which bracket the discrepancies are greatest, the exact and approximate curves do not follow the same form of variation as time increases above five seconds. This can be seen by an inspection of the defining equations. Whereas for large values of time the approximate solution gives $V_A^2 + V_B^2$ of the form $(at)^{-n}$, the exact solution gives a form $b(at)^{-n}$ with a decreasing coefficient sinusoid added to it.

For $\tau_1 = 1$ and t large:

Approximate: $V_A^2 + V_B^2 \cong \frac{1}{t^2}$

Exact: $V_A^2 + V_B^2 \cong \frac{1.5}{t^2} + \frac{\sin t - \cos t}{t^2}$

For $\tau_1 = \frac{1}{2}$ and t large:

Approximate: $V_A^2 + V_B^2 \cong \frac{1}{(.5t)^2}$

Exact: $V_A^2 + V_B^2 \cong \frac{1}{(.5t)^2} - \frac{\cos t}{(.5t)^3}$

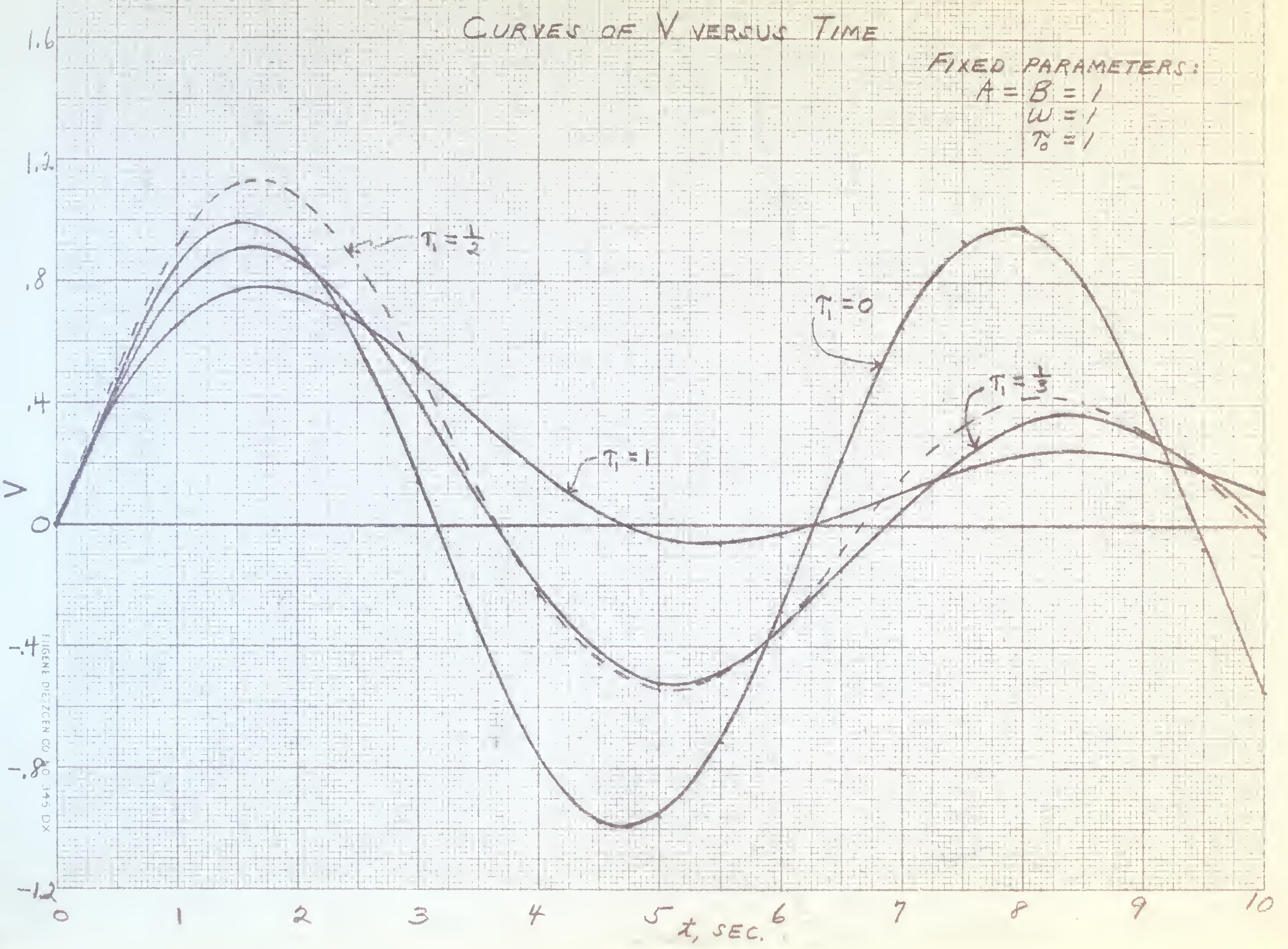
Though for large values of t the error may hence not be great in using the approximate solution the percentage error may be very great.

COMPLETE SOLUTION FOR CASE II

T_1	GENERAL SOLUTION
1	$V=\left[\frac{T_0(A-\omega T_0 B)}{\omega^2T_0^2+1}+\frac{B}{\omega}+\frac{A}{\omega}\sin\omega t-\frac{B}{\omega}\cos\omega t\right]\frac{1}{T_0+T_1 t}$
$\frac{1}{2}$	$V=\left[\frac{T_0^2(A-\omega T_0 B)}{\omega^2T_0^2+1}+\frac{BT_0}{\omega}-\frac{AT_1}{\omega^2}+\left(\frac{AT_0}{\omega}+\frac{AT_1t}{\omega}+\frac{BT_1}{\omega^2}\right)\sin\omega t+\left(\frac{AT_1}{\omega^2}-\frac{BT_0}{\omega}-\frac{BT_1t}{\omega}\right)\cos\omega t\right](T_0+T_1t)^{-2}$
$\frac{1}{3}$	$V=\left[\frac{T_0^3(A-\omega T_0 B)}{\omega^2T_0^2+1}+\frac{BT_0^2}{\omega}-\frac{2AT_0T_1}{\omega^2}-\frac{2BT_1^2}{\omega^3}+\left(\frac{AT_0^2}{\omega}+\frac{2AT_0T_1t}{\omega}+\frac{AT_1^2t^2}{\omega}-\frac{2AT_1^2}{\omega^3}+\frac{2BT_0T_1}{\omega^2}+\frac{2BT_1^2t}{\omega^2}\right)\sin\omega t+\left(\frac{2AT_0T_1}{\omega^2}+\frac{2AT_1^2t}{\omega^2}-\frac{BT_0^2}{\omega}-\frac{2BT_0T_1t}{\omega}-\frac{BT_1^2t^2}{\omega}+\frac{2BT_1^2}{\omega^3}\right)\cos\omega t\right](T_0+T_1t)^{-3}$
$\frac{1}{4}$	$V=\left[\frac{T_0^4(A-\omega T_0 B)}{\omega^2T_0^2+1}+\frac{6T_1^3A}{\omega^4}+\frac{BT_0^3}{\omega}-\frac{3T_0^2T_1A}{\omega^2}-\frac{6T_0T_1^2B}{\omega^3}+\left(\frac{AT_0^3}{\omega}+\frac{3AT_0^2T_1t}{\omega}+\frac{3AT_0T_1^2t^2}{\omega}-\frac{6T_0T_1^2A}{\omega^3}+\frac{T_1^3At^3}{\omega}-\frac{6T_1^3At}{\omega^3}+\frac{3T_0^2T_1B}{\omega^2}+\frac{6T_0T_1^2Bt}{\omega^2}+\frac{3T_1^3Bt^2}{\omega^2}-\frac{6T_1^3B}{\omega^4}\right)\sin\omega t+\left(\frac{3T_0^2T_1A}{\omega^2}+\frac{6T_0T_1^2At}{\omega^2}+\frac{3T_1^3At^2}{\omega^2}-\frac{6T_1^3A}{\omega^4}-\frac{BT_0^3}{\omega}-\frac{3T_0^2T_1Bt}{\omega}-\frac{3T_0T_1^2Bt^2}{\omega}+\frac{6T_0T_1^2B}{\omega^3}-\frac{T_1^3Bt^3}{\omega}+\frac{6T_1^3Bt}{\omega^3}\right)\cos\omega t\right](T_0+T_1t)^{-4}$

T_1	SOLUTION FOR $A = 0$
1	$V = \left[\frac{-\omega T_0^2 B}{\omega^2 T_0^2 + 1} + \frac{B}{\omega} - \frac{B}{\omega} \cos \omega t\right] (T_0 + T_1 t)^{-1}$
$\frac{1}{2}$	$V = \left[\frac{-\omega T_0^3 B}{\omega^2 T_0^2 + 1} + \frac{B T_0}{\omega} + \frac{B T_1}{\omega^2} \sin \omega t - \left(\frac{B T_0}{\omega} + \frac{B T_1 t}{\omega}\right) \cos \omega t\right] (T_0 + T_1 t)^{-2}$
$\frac{1}{3}$	$V = \Big[\frac{-\omega T_0^4 B}{\omega^2 T_0^2 + 1} + \frac{B T_0^2}{\omega} - \frac{2 B T_1^2}{\omega^3} + \left(\frac{2 B T_0 T_1}{\omega^2} + \frac{2 B T_1^2 t}{\omega^2}\right) \sin \omega t$ $- \left(\frac{B T_0^2}{\omega} + \frac{2 B T_0 T_1 t}{\omega} + \frac{B T_1^2 t^2}{\omega} - \frac{2 B T_1^2}{\omega^3}\right) \cos \omega t\Big] (T_0 + T_1 t)^{-3}$
$\frac{1}{4}$	$V = \Big[\frac{-\omega T_0^5 B}{\omega^2 T_0^2 + 1} + \frac{B T_0^3}{\omega} - \frac{6 T_0 T_1^2 B}{\omega^3} + \left(\frac{3 T_0^2 T_1 B}{\omega^2} + \frac{6 T_0 T_1^2 B t}{\omega^2} + \frac{3 T_1^3 B t^2}{\omega^2} - \frac{6 T_1^3 B}{\omega^4}\right) \sin \omega t$ $- \left(\frac{B T_0^3}{\omega} + \frac{3 T_0^2 T_1 B t}{\omega} + \frac{3 T_0 T_1^2 B t^2}{\omega} + \frac{T_1^3 B t^3}{\omega} - \frac{6 T_0 T_1^2 B}{\omega^3} - \frac{6 T_1^3 B t}{\omega^3}\right) \cos \omega t\Big] (T_0 + T_1 t)^{-4}$

T	Solution for V =
1	$V = \left[\frac{T_0 A}{\omega^2 T_0^2+1} + \frac{A}{\omega}\sin\omega t\right](T_0+T_1 t)^{-1}$
$\frac{1}{2}$	$V = \left[\frac{T_0^2 A}{\omega^2 T_0^2+1} - \frac{A T_1}{\omega^2} + \left(\frac{A T_0}{\omega} + \frac{A T_1 t}{\omega}\right)\sin\omega t + \frac{A T_1}{\omega^2}\cos\omega t\right](T_0+T_1 t)^{-2}$
$\frac{1}{3}$	$V = \left[\frac{T_0^3 A}{\omega^2 T_0^2+1} - \frac{2AT_0T_1}{\omega^2} + \left(\frac{AT_0^2}{\omega} + \frac{2AT_0T_1t}{\omega} + \frac{AT_1^2t^2}{\omega} - \frac{2AT_1^2}{\omega^3}\right)\sin\omega t$ $+ \left(\frac{2AT_0T_1}{\omega^2} + \frac{2AT_1^2t}{\omega^2}\right)\cos\omega t\right](T_0+T_1 t)^{-3}$
$\frac{1}{4}$	$V = \left[\frac{T_0^4 A}{\omega^2 T_0^2+1} + \frac{6T_1^3 A}{\omega^4} - \frac{3T_0^2T_1A}{\omega^2} + \left(\frac{AT_0^3}{\omega} + \frac{3T_0^2T_1At}{\omega} + \frac{3T_0T_1^2At^2}{\omega} - \frac{6T_0T_1^2A}{\omega^3}\right.\right.$ $\left.\left.+ \frac{T_1^3At^3}{\omega} - \frac{6T_1^3At}{\omega^3}\right)\sin\omega t + \left(\frac{3T_0^2T_1A}{\omega^2} + \frac{6T_0T_1^2At}{\omega^2} + \frac{3T_1^3At^2}{\omega^2} - \frac{6T_1^3A}{\omega^4}\right)\cos\omega t\right](T_0+T_1 t)^{-4}$



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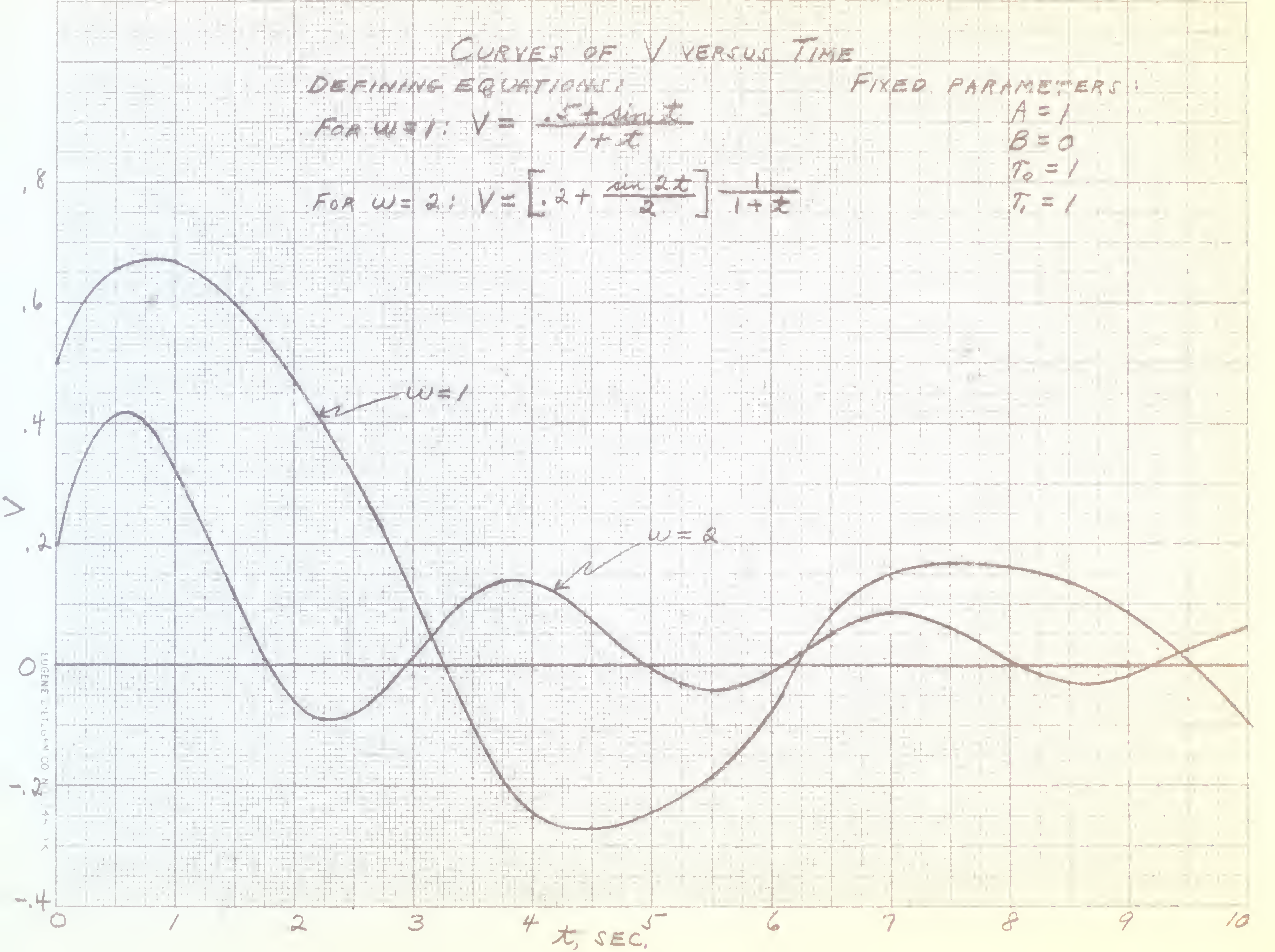


CURVES OF V VERSUS TIME

DEFINING EQUATIONS:

FOR $w = 1$: $V = \frac{.5 + \sin t}{1 + t}$

FOR $w = 2$: $V = \left[.2 + \frac{\sin 2t}{2}\right] \frac{1}{1 + t}$

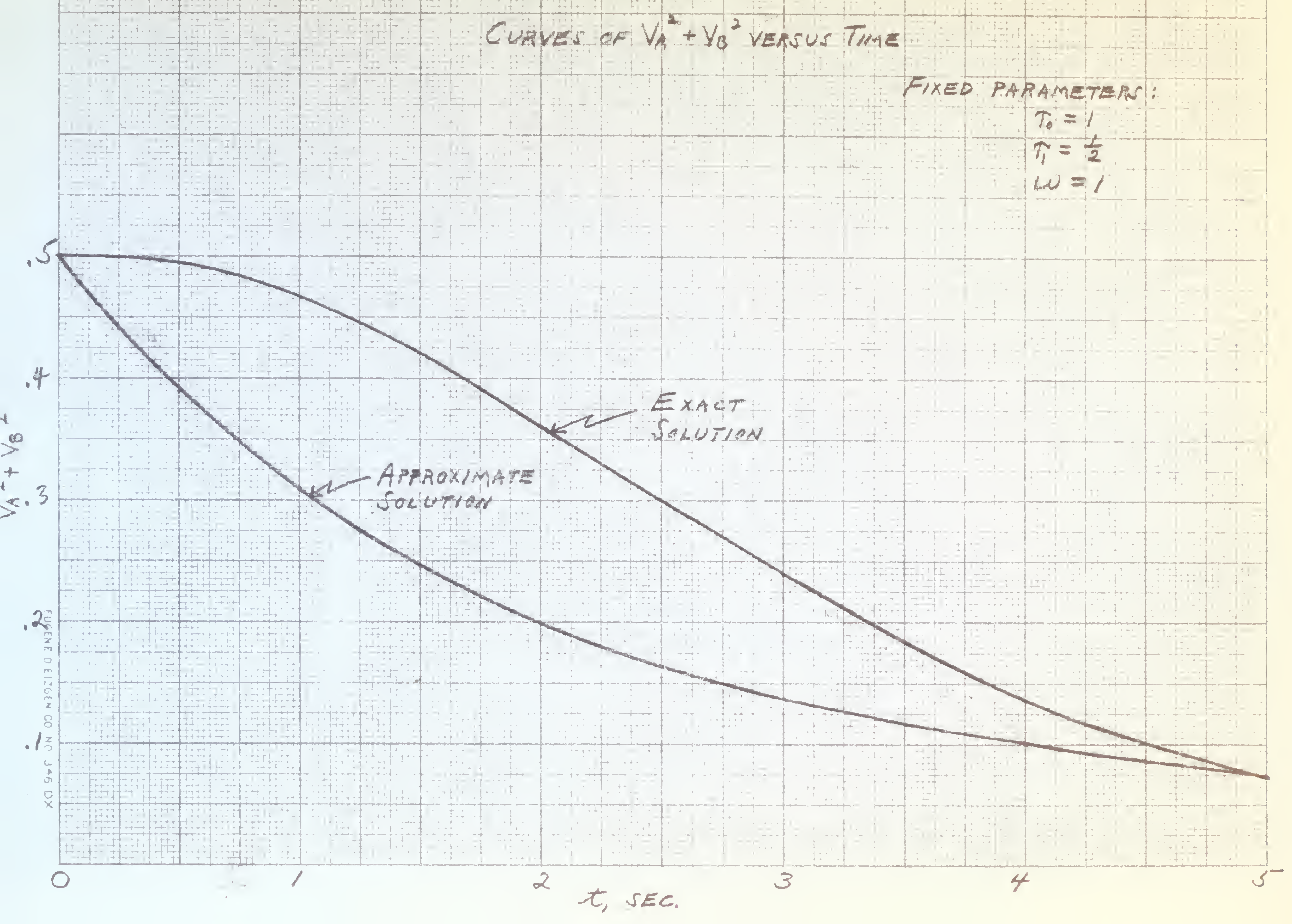
FIXED PARAMETERS:

$A = 1$

$B = 0$

$T_0 = 1$

$T_1 = 1$



CURVES OF $V_A^2 + V_B^2$ VERSUS TIME

FIXED PARAMETERS:
$T_0 = 1$
$T_1 = \frac{1}{2}$
$\omega = 1$

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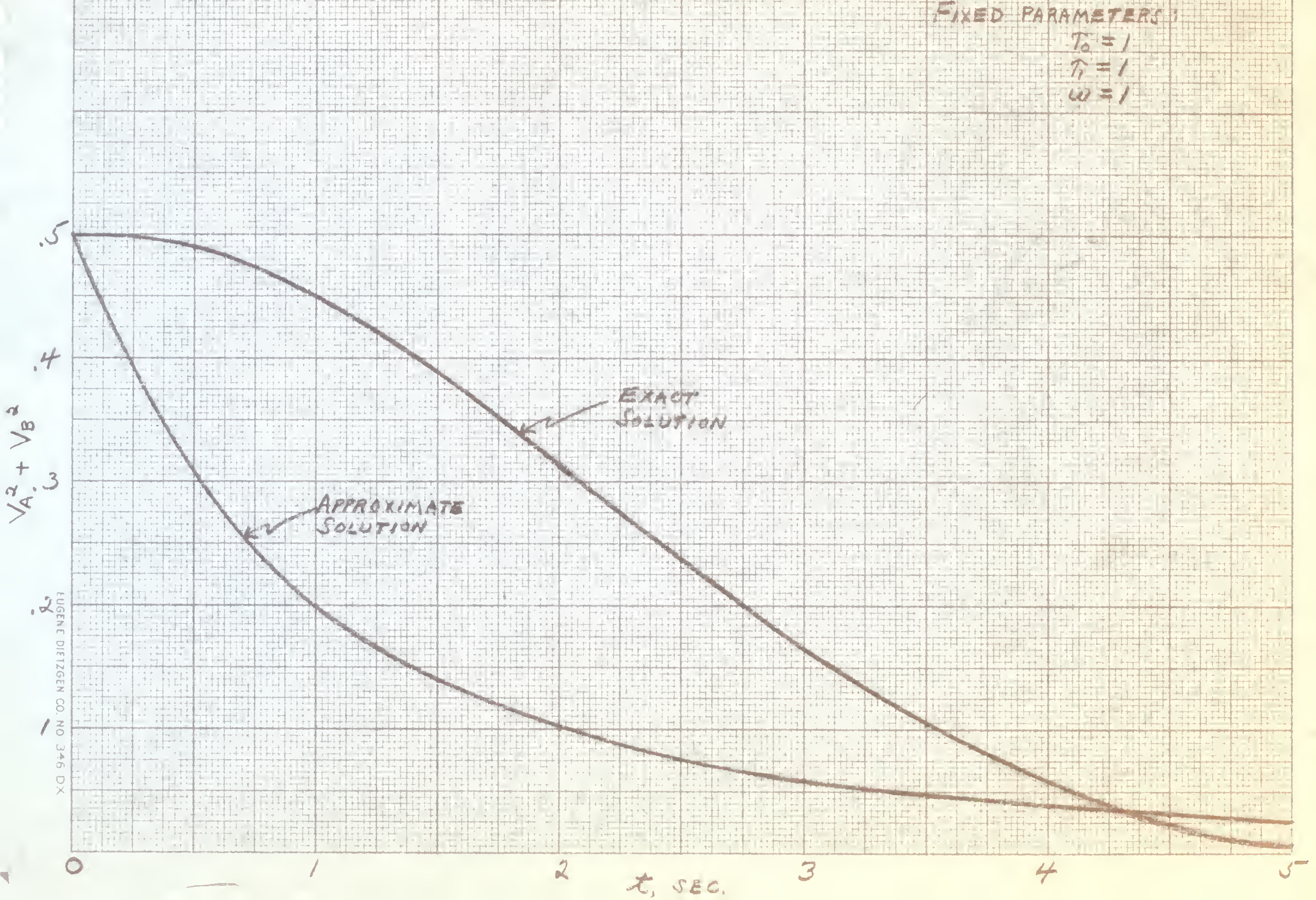
Curves of $V_A^2 + V_B^2$ Versus Time



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Appendix

Classical methods were preferred to the method of Laplace Transformations in the solutions to the differential equations, since the transforms corresponding to the varying circuit parameters are not generally available in the literature.

As an example the formal solution for V(t) corresponding to a parabolic signal will be presented here:

Equation (A) should perhaps more accurately read:

$$(8)\quad (\tau_0 + \tau_1 t)\frac{dV(t)}{dt} + V(t) = e_0 + e_1 t + e_2 t^2 + 0$$

Because of the linearity of this equation one may solve for V(t) due to the right hand side being 0 and $e_0 + e_1 t + e_2 t^2$ separately. These two parts of the solution are commonly termed the complementary function or transient and particular integral or steady state solution.

For the transient:

$$(8a)\quad (\tau_0 + \tau_1 t)\frac{dV(t)}{dt} + V(t) = 0$$

$$\text{or}\quad \frac{dV(t)}{V(t)} = -\frac{dt}{\tau_0 + \tau_1 t} = -\frac{1}{\tau_1}\,\frac{dt}{t + \frac{\tau_0}{\tau_1}}$$

Integrating

$$\log V(t) = -\frac{1}{\tau_1}\log\left[\frac{\tau_0}{\tau_1} + t\right] + \log K = \log\frac{K}{\left[\frac{\tau_0}{\tau_1} + t\right]^{\frac{1}{\tau_1}}}$$

where K is an arbitrary constant.

Consequently

$$(9a)\quad V(t)_{\text{transient}} = \frac{K}{\left[\frac{\tau_0}{\tau_1} + t\right]^{\frac{1}{\tau_1}}}$$

For the steady state solution

Assume that

$$(8b)\quad V(t) = V_0 + V_1 t + V_2 t^2$$

i.e. V(t) is assumed to be expressible in a power series terminating in the same degree as e(t)

Differentiating (8b)

$$(8c)\quad \frac{dV(t)}{dt} = V_1 + 2V_2 t$$

Multiplying (8c) by $(\tau_0 + \tau_1 t)$ and adding to (8b) there results

$$(8d)\quad (\tau_0 + \tau_1 t)(V_1 + 2V_2 t) + V_0 + V_1 t + V_2 t^2$$

$$= \left[V_0 + V_1\tau_0\right] + \left[V_1(1+\tau_1) + 2\tau_0 V_2\right] t + V_2(1 + 2\tau_1)\, t^2$$

$$= e_0 + e_1 t + e_2 t^2 \text{ for } \underline{\underline{\text{all}}} \text{ instants of time by (8)}$$

Therefore:

$$(9b)\quad \begin{cases} V_0 + \tau_0 V_1 = e_0 \\ V_1(1 + \tau_1) + 2\tau_0 V_2 = e_1 \\ V_2(1 + 2\tau_1) = e_2 \end{cases}$$

Solving (9b) simultaneously

$$\left.\begin{aligned} V_2 &= \frac{e_2}{1+2\tau_1} \\ V_1 &= \frac{e_1}{1+\tau_1} - \frac{2\tau_o^2 e_2}{(1+\tau_1)(1+2\tau_1)} \\ V_o &= e_1 \quad \frac{-\tau_o e_1}{1+\tau_1} + \frac{2\tau_o^2 e_2}{(1+\tau_1)(1+2\tau_1)} \end{aligned}\right\} \qquad (9c)$$

The complete solution is:

$$(9) \quad V(t) = K\left[\frac{1}{\frac{\tau_o}{\tau_1}+t}\right]^{\frac{1}{\tau_1}} + \left[e_o \quad \frac{-\tau_o e_1}{1+\tau_1} + \frac{2\tau_o^2 e_2}{(1+\tau_1)(1+2\tau_1)}\right]$$

$$+ \left[\frac{e_1}{1+\tau_1} - \frac{2\tau_o e_2}{(1+\tau_1)(1+2\tau_1)}\right] t + \frac{e_2}{1+2\tau_1} \quad t^2$$

For t = 0

$$(9d) \quad V(0) = K\left[\frac{\tau_1}{\tau_o}\right]^{\frac{1}{\tau_1}} + \left[e_o \quad \frac{-\tau_o e_1}{1+\tau_1} + \frac{2\tau_o^2 e_2}{(1+\tau_1)(1+2\tau_1)}\right] = \text{by the boundary condition.}$$

Solving (9d) for K

$$(9e) \qquad K = \left[\frac{\tau_o e_1}{1+\tau_1} - \frac{2\tau_o^2 e_2}{(1+\tau_1)(1+2\tau_1)}\right]\left[\frac{\tau_o}{\tau_1}\right]^{\frac{1}{\tau_1}}$$

The method of "assuming" the steady state solution may be employed to obtain solutions to any number of finite terms appearing in the expansion for e (t), i.e. let

$$e\,(t) = e_o + e_1 t + e_2 t^2 \; - - - - - - \; e_n t^n \tag{10}$$

Assume as a solution to equation (1) for $CR(t) = \tau_o + \tau_1 t$

$$V\,(t) = V_o + V_1 t + V_2 t^2 + - - - - - - - - \; V_n t^n \tag{11}$$

By the previous method one obtains as solutions

$$\left\{\begin{aligned}
V_n &= \frac{e_n}{1 + n\tau_1} \\
V_{n-1} &= \frac{e_{n-1}}{1 + (n-1)\tau_1} - \frac{n\tau_o e_n}{(1 + n\tau_1)(1 + [n-1]\tau_1)} \\
V_{n-2} &= \frac{e_{n-2}}{1 + (n-2)\tau_1} - \frac{(n-1)\tau_o e_{n-1}}{[1 + (n-1)\tau_1][1 + (n-2)\tau_1]} + \frac{n\,(n-1)\tau_o^2 e_n}{(1 + n\tau_1)(1 + [n-1]\tau_1)(1[n-2]\tau_1)} \\
&\;\vdots \\
V_o &= \frac{(-1)^n n!\,\tau_o^n e_n}{(1 + n\tau_1)(1 + [n-1]\tau_1)(1 + [n-2]\tau_1) - - - (1 + \tau_1)} \\
&\quad \frac{(-1)^{n-1} (n-1)!\,\tau_o^{n-1} e_{n-1}}{(1 + [n-1]\tau_1)(1 + [n-2]\tau_1) - - - - (1 + \tau_1)} + - - - - \frac{\tau_o e_1}{1 + \tau_1} + e_o
\end{aligned}\right. \tag{12}$$

This scheme of "assuming" the steady state solution will succeed as long as the resistance is constant or varies linearly with time; however, as soon as parabolic or higher order terms are admitted into the expansion of R(t) the method fails. The reason for failure is easily evident: $\frac{dV(t)}{dt}$ is an expansion whose highest degree term is of one degree lower than the highest degree term of V (t). Multiplying $\frac{dV(t)}{dt}$ by a term containing t^2 or t raised to a power higher than 2 will cause the degree of the left hand side of equation (1) to exceed the degree of the right hand side, making it impossible to equate coefficients.

However, since equation (1) is a first order linear differential equation its solution can always be placed in integral form if not closed form. For our equation:

$$CR(t)\frac{dV(t)}{dt}+V(t)=e(t) \tag{13}$$

$$\frac{dV(t)}{dt}+\frac{V(t)}{CR(t)}=\frac{e(t)}{CR(t)} \tag{14}$$

For the homogeneous equation (right hand side = 0), we have:

$$\frac{dV(t)}{dt}+\frac{V(t)}{CR(t)}=0 \tag{15}$$

Separating the variables we obtain:

$$\frac{dV}{V}=-\frac{dt}{CR(t)}$$

$$\int\frac{dV}{V}=-\int\frac{dt}{CR(t)}$$

$$\log\frac{V}{K_1}=-\int\frac{dt}{CR(t)}$$

$$V=K_1e^{-\int\frac{dt}{CR(t)}} \tag{16}$$

When the right hand side of the equation is not equal to zero, we assume the same form of solution but in this case the constant K_1 is replaced by u, where u is now a time function.

Then:

$$V=ue^{-\int\frac{dt}{CR(t)}} \tag{17}$$

$$\frac{dV}{dt}=\frac{du}{dt}e^{-\int\frac{dt}{CR(t)}}-\frac{u}{CR(t)}e^{-\int\frac{dt}{CR(t)}}$$

$$\frac{dV}{dt}=\left[\frac{du}{dt}-\frac{u}{CR(t)}\right]e^{-\int\frac{dt}{CR(t)}}$$

Substituting into the original equation we have:

$$\left[\frac{du}{dt} - \frac{u}{CR(t)}\right] e^{-\int\frac{dt}{CR(t)}} + \frac{u}{CR(t)} e^{-\int\frac{dt}{CR(t)}} = \frac{e(t)}{CR(t)}$$

so that:

$$\frac{du}{dt} e^{-\int\frac{dt}{CR(t)}} = \frac{e(t)}{CR(t)}$$

Then:

$$\frac{du}{dt} = \frac{e(t)}{CR(t)} e^{\int\frac{dt}{CR(t)}}$$

$$u = \int \frac{e(t)}{CR(t)} e^{\int\frac{dt}{CR(t)}} dt + K_2$$

Therefore:

$$V = ue^{-\int\frac{dt}{CR(t)}}$$

$$V = \left[K_2 + \int \frac{e(t)}{CR(t)} e^{\int\frac{dt}{CR(t)}} dt\right] e^{-\int\frac{dt}{CR(t)}} \tag{18}$$

This method of attack, involving the solution of an integral equation, is used in this paper when (1) R(t) is assumed to consist of terms giving rise to a variation greater than linear, and (2) the signal voltages are of sinusoidal form.

For R(t) consisting of terms of higher order than linear subsequent analysis shows that the "transient" part of the solution can be placed into closed form, but the steady state solution cannot. However, an integral equation in itself constitutes appreciable progress compared to the differential equation form.

$$(19)\ \left\{\text{Consider}\quad \begin{aligned} R(t) &= R_0 + R_1 t + R_2 t^2 \\ CR(t) &= \tau_0 + \tau_1 t + \tau_2 t^2 = \tau_2\left(t^2 + \frac{\tau_1}{\tau_2} t + \frac{\tau_0}{\tau_2}\right) \end{aligned}\right\}$$

The general solution for V(t) corresponding to e (t)

$= e_0 + e_1 t + e_2 t^2 + \text{- - - - - - - - -} \; e_n t^n$ is by (18)

$$(20)\quad V(t) = e^{-\int \frac{dt}{T_2\left(t^2+\frac{T_1}{T_2}t+\frac{T_0}{T_2}\right)}} \int e^{\int \frac{dt}{T_2\left(t^2+\frac{T_1}{T_2}t+\frac{T_0}{T_2}\right)}} \left[\frac{e_0+e_1t+\dots e_nt^n}{T_2\left(t^2+\frac{T_1}{T_2}t+\frac{T_0}{T_2}\right)}\right]dt + K e^{-\int \frac{dt}{T_2\left(t^2+\frac{T_1}{T_2}t+\frac{T_0}{T_2}\right)}}$$

The expression $e^{-\int \frac{dt}{T_2\left(t^2+\frac{T_1}{T_2}t+\frac{T_0}{T_2}\right)}}$ is to be found. Three distinct subcases arise:

$$1-\quad \left(t^2+\frac{T_1}{T_2}t+\frac{T_0}{T_2}\right) = (t+\alpha)^2 \qquad \int\frac{dt}{(t+\alpha)^2} = -\frac{1}{t+\alpha}$$

and

$$(20\text{ a})\quad V(t) = e^{\frac{1}{T_2(t+\alpha)}}\left[K-\int(e_1+2e_2t+\dots ne_nt^{n-1})\,e^{-\frac{1}{T_2(t+\alpha)}}dt\right] + e_0+e_1t+e_2t^2+\dots e_nt^n$$

$$2-\quad \left(t^2+\frac{T_1}{T_2}t+\frac{T_0}{T_2}\right) = (t+\alpha)^2+\beta^2 = \beta^2\left[\frac{(t+\alpha)^2}{\beta^2}+1\right]$$

$$\int\frac{dt}{t^2+\frac{T_1}{T_2}t+\frac{T_0}{T_2}} = \frac{1}{\beta}\tan^{-1}\frac{t+\alpha}{\beta}$$

and

$$(20\text{ b})\quad V(t) = e^{-\frac{1}{\beta T_2}\tan^{-1}\frac{t+\alpha}{\beta}}\left[K-\int(e_1+2e_2t+\dots ne_nt^{n-1})\,e^{\frac{1}{\beta T_2}\tan^{-1}\frac{t+\alpha}{\beta}}dt\right] + e_0+e_1t+e_2t^2+\dots+e_nt^n$$

$\Gamma_2\left(t^2+\frac{\Gamma_1}{\Gamma_2}t+\frac{\Gamma_0}{\Gamma_2}\right) = (t+\alpha)^2-\beta^2 =$ [illegible]

[illegible]

and

(20 c) $V(t) = \left[\frac{t+\alpha+\beta}{t+\alpha-\beta}\right]^{\frac{1}{2\beta\Gamma_2}}$ [illegible]

It is interesting to compare the type of transients obtained:

C R (t)	Transient	Limit of Transient as t → ∞
Γ_0	$K e^{-\frac{t}{\Gamma_0}}$	0
$\Gamma_0+\Gamma_1 t$	$K(\Gamma_0+\Gamma_1 t)^{-\frac{1}{\Gamma_1}}$	0
$\Gamma_0+\Gamma_1 t+\Gamma_2 t^2$	$K e^{\frac{1}{t+\alpha}}$	K
	$K e^{-\frac{1}{\beta\Gamma_2}\tan^{-1}\frac{t+\alpha}{\beta}}$	$K e^{-\frac{\pi}{2\beta\Gamma_2}}$
	$K\left(\frac{t+\alpha+\beta}{t+\alpha-\beta}\right)^{\frac{1}{2\beta\Gamma_2}}$	K

The last three transients approach finite values. If C R (t) contains higher than second order terms it is necessary to factor $\Gamma_0+\Gamma_1 t - - - \Gamma_m t^m$ into terms of the type $(t+\alpha)^k$ $\left[(t+\alpha)^2+\beta^2\right]^l$ and $\left[(t+\alpha)^2-\beta^2\right]^p$ (k, l, p are integers equal to or greater than 1). One can obtain the proper transient solutions similar to the ones corresponding to C R (t) = $\Gamma_0+\Gamma_1 t+\Gamma_2 t^2$ by a partial fractions expansion.

In so far as sinusoidal signal voltages are concerned, to illustrate the method of obtaining the results tabulated previously in the paper, the following sample calculations are shown for the case in which $C R_1 = \tau_1 = 1$.

$$\frac{dV}{dt} + \frac{V}{CR(t)} = \frac{e(t)}{CR(t)}$$

(21) Let $e(t) = A \cos u t + B \sin u t$

$$CR(t) = \tau_o + \tau_1 t$$

$$\frac{dV}{dt} + \frac{V}{\tau_o + \tau_1 t} = \frac{A \cos u t + B \sin u t}{\tau_o + \tau_1 t}$$

Then:

$$V = \left[K_2 + \int \frac{A \cos u t \quad B \sin u t}{\tau_o + \tau_1 t} e^{\int \frac{dt}{\tau_o + \tau_1 t}} dt \right] e^{-\int \frac{dt}{\tau_o + \tau_1 t}}$$

$$V = \left[K_2 + \int \frac{A \cos u t \quad B \sin u t}{\tau_o + \tau_1 t} e^{\int \frac{1}{\tau_1} \log (\tau_o + \tau_1 t)} dt \right] e^{-\frac{1}{\tau_1} \log (\tau_o + \tau_1 t)}$$

(22) $$V = \left[K_2 + \int (A \cos u t + B \sin u t)(\tau_o + \tau_1 t)^{\frac{1}{\tau_1} - 1} dt \right] (\tau_o + \tau_1 t)^{-\frac{1}{\tau_1}}$$

Since $\tau_1 = 1$:

$$V = \left[K_2 + \int (A \cos u t + B \sin u t) \, dt \right] \frac{1}{\tau_o + t}$$

(23) $$V = \left[K_3 + \frac{A}{u} \sin u t - \frac{B}{u} \cos u t \right] \frac{1}{\tau_o + t}$$

To evaluate the arbitrary constant K_3 we must go back to the steady state solution for R constant and substitute $t = 0$, giving us $V(0)$, our boundary condition for the variable resistance solution above.

For constant resistance we have:

(24) $$\frac{dV}{dt} + \frac{V}{\tau_o} = \frac{A \cos u t + B \sin u t}{\tau_o}$$

The steady state solution is found as follows:

Let V_2 = steady state solution for V

Let $V_2 = M \cos u\, t + N \sin u\, t$

$$\frac{dV_2}{dt} = - u\, M \sin u\, t + u\, N \cos u\, t$$

$$- u\, M \sin u\, t + u\, N \cos u\, t + \frac{M}{\tau_o} \cos u\, t + \frac{N}{\tau_o} \sin u\, t = \frac{A}{\tau_o} \cos u\, t + \frac{B}{\tau_o} \sin u\, t$$

Equating sine terms and cosine terms separately:

$$\begin{cases} - u\, M + \dfrac{N}{\tau_o} = \dfrac{B}{\tau_o} \\ u\, N + \dfrac{M}{\tau_o} = \dfrac{A}{\tau_o} \end{cases}$$

$$\begin{cases} - u\tau_o\, M + N = B \\ u\tau_o\, N + M = A \end{cases}$$

$$- u\tau_o\, M + N = B$$

$$u\tau_o\, M + u^2\tau_o^{\ 2} N = u\tau_o\, A$$

$$N\,(1 + u^2\tau_o^{\ 2}) = u\,\tau_o A + B$$

$$N = \frac{u\tau_o\, A + B}{u^2\tau_o^{\ 2} + 1}$$

$$M = A - u\tau_o\, N = A - \frac{u\tau_o\,(u\tau_o\, A + B)}{u^2\tau_o^{\ 2} + 1} = \frac{A - u\tau_o\, B}{u^2\tau_o^{\ 2} + 1}$$

(25) $$V_2 = \frac{A - u\tau_o\, B}{u^2\tau_o^{\ 2} + 1} \cos u\, t + \frac{u\tau_o\, A + B}{u^2\tau_o^{\ 2} + 1} \sin u\, t$$

For the variable resistance case, $V(0) = V_2\,(0)$.

(26) $$V_2(0) = \frac{A - u\tau_o\, B}{u^2\tau_o^{\ 2} + 1}$$

$$V\ (0) = \frac{A - u\tau_o B}{u^2\tau_o^2 + 1} = \left[K_3 - \frac{B}{u}\right] \frac{1}{\tau_o}$$

$$(27)\quad K_3 = \frac{\tau_o\ (A - u\tau_o\ B)}{u^2\tau_o^2 + 1} + \frac{B}{u}$$

The complete solution for V in the variable resistance case is then:

$$(28)\quad V = \left[\frac{{}_o\ (A - u\tau_o\ B)}{u^2\tau_o^2 + 1} + \frac{B}{u} + \frac{A}{u}\sin u\ t - \frac{B}{u}\cos u\ t\right] \cdot \frac{1}{\tau_o + \tau_1\ t}$$

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